

Mapping class groups are not linear in positive characteristic

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Abstract

For Σ an orientable surface of finite topological type having genus at least 3 (possibly closed or possibly with any number of punctures or boundary components), we show that the mapping class group $Mod(\Sigma)$ has no faithful linear representation in any dimension over any field of positive characteristic.

1 Introduction

A common question to ask of a given infinite finitely generated group is whether it is linear. For instance consider the braid groups B_n , the automorphism group $Aut(F_n)$ of the free group F_n and the mapping class group $Mod(\Sigma_g)$ of the closed orientable surface Σ_g with genus g . Linearity in the first case was open for a while but is now known to hold by [1], [8]. For $n \geq 3$ [5] showed that $Aut(F_n)$ is not linear, whereas for $g \geq 3$ the third case is open. However whereas the definition of linearity is that a group embeds in $GL(d, \mathbb{F})$ for some $d \in \mathbb{N}$ and some field \mathbb{F} , in practice one tends to concentrate on the case where $\mathbb{F} = \mathbb{C}$. In fact a finitely generated group embeds in \mathbb{C} if and only if it embeds in some field of characteristic zero, so it is enough to restrict to this case if only characteristic zero representations are being considered.

However we can still ask about faithful linear representations in positive characteristic. For instance in the three examples above, it is unknown for $n \geq 4$ if the braid group B_n admits a faithful linear representation in any dimension over any field of positive characteristic. For $Aut(F_n)$ with $n \geq 3$, the proof in [5] applies to any field, not just the characteristic zero case, so that there are also no faithful representations in positive characteristic.

As for mapping class groups, we show here that there are no faithful linear representations of $Mod(\Sigma_g)$ in any dimension over any field of positive characteristic when Σ_g is an orientable surface of finite topological type having genus g at least 3 (which might be closed or might have any number of punctures or boundary components).

The idea comes from considering the analogy between a finitely generated group being linear in positive characteristic and having a “nice” geometric action, as we did in [4] when showing that Gersten’s free by cyclic group has no faithful linear representation in any positive characteristic. On looking more closely to see which definition of “nice” aligns most closely with linearity in positive characteristic, we were struck by the similarities between that and the notion of a finitely generated group acting properly and semisimply (more so than properly and cocompactly) on a complete CAT(0) space. In [3] Bridson shows that for all the surfaces Σ_g mentioned above, the mapping class group $Mod(\Sigma_g)$ does not admit such an action. This result is first credited to [6] but the proof in [3] consists of finding an obstruction to the existence of such an action by any one of these groups. This obstruction involves taking an element of infinite order and its centraliser in said group, then applying a condition on the abelianisation of this centraliser. Here we show that this condition holds verbatim for groups which are linear in positive characteristic, thus obtaining the same obstruction.

We leave open the question of whether the mapping class group of the closed orientable surface of genus 2 is linear in positive characteristic, but we note that it was shown in [2] and [7] using the braid group results that this group is linear in characteristic zero anyway.

2 Proof

The following is the crucial point which distinguishes our treatment of linear groups in positive characteristic from the classical case.

Proposition 1 *If \mathbb{F} is an algebraically closed field of positive characteristic and $d \in \mathbb{N}$ then there exists $K \in \mathbb{N}$ such that for all elements $g \in GL(d, \mathbb{F})$ the matrix g^K is diagonalisable.*

Proof. If \mathbb{F} has characteristic p then we take K to be any power of p which is at least d .

We put g into Jordan normal form, or indeed any form where the matrix splits up into blocks corresponding to the generalised eigenspaces of g and such that we are upper triangular in each block. Then on taking the eigenvalue $\lambda \in \mathbb{F}$ of g , the block of g corresponding to λ will be of the form $\lambda I + N$ where N is upper triangular with all zeros on the diagonal so that

$$(\lambda I + N)^K = \lambda^K I + \binom{K}{1} \lambda^{K-1} N + \dots + \binom{K}{K-1} \lambda N^{K-1} + N^K.$$

But $N^K = 0$ because $K \geq d$ and $\binom{K}{i} \equiv 0 \pmod{p}$ for $0 < i < K$ as K is a power of p . Thus in this block we have that g^K is equal to $\lambda^K I$. But we can do this in each block, making g^K a diagonal matrix. \square

As for the mapping class group $\text{Mod}(\Sigma)$ of the surface Σ , we have:

Proposition 2 ([3] Proposition 4.2)

If Σ is an orientable surface of finite type having genus at least 3 (with any number of boundary components and punctures) and if T is the Dehn twist about any simple closed curve in Σ then the abelianisation of the centraliser in $\text{Mod}(\Sigma)$ of T is finite.

This is in contrast to:

Theorem 3 *Suppose that G is a linear group over a field of positive characteristic and $C_G(g)$ is the centraliser in G of the infinite order element g . Then the image of g in the abelianisation of $C_G(g)$ also has infinite order.*

Proof. As the abelianisation is the universal abelian quotient of a group, it is enough to find some homomorphism of $C_G(g)$ to an abelian group where g maps to an element of infinite order, so we use the determinant.

We first replace our field by its algebraic closure. Then Proposition 1 tells us that we have the diagonalisable element g^K , whereupon showing that g^K has infinite order in the abelianisation of $C_G(g)$ (which could of course be smaller than the centraliser in G of g^K) will establish the same for g .

Take a basis so that g^K is actually diagonal and group together repeated eigenvalues, so that we have

$$g^K = \begin{pmatrix} \boxed{\lambda_1 I_{d_1}} & & 0 \\ & \ddots & \\ 0 & & \boxed{\lambda_k I_{d_k}} \end{pmatrix}.$$

This means that any element in $C_G(g^K)$, and thus also in $C_G(g)$, is of the form

$$\begin{pmatrix} \boxed{A_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{A_k} \end{pmatrix}$$

with the same block structure.

Consequently we have as homomorphisms from $C_G(g)$ to the multiplicative abelian group (\mathbb{F}^*, \times) not just the determinant itself but also the “sub-determinant” functions \det_1, \dots, \det_k . Here for $h \in C_G(g)$ we define $\det_i(h)$ as the determinant of the i th block of h when expressed with respect to our basis above which diagonalises g^K , and this is indeed a homomorphism.

Now it could be that $\det_i(g^K)$ has finite order, which implies that $\lambda_i^{d_i}$ and thus also λ_i has finite multiplicative order in \mathbb{F}^* . However if this is true for all $i = 1, 2, \dots, k$ then $\lambda_1, \dots, \lambda_k$ all have finite order. This means that g^K and so g does too, which is a contradiction. Thus for some i we know g^K and g map to elements of infinite order in the abelian group (\mathbb{F}^*, \times) under the homomorphism \det_i .

□

Corollary 4 *If Σ is an orientable surface of finite type having genus at least 3 (with any number of boundary components and punctures) then $\text{Mod}(\Sigma)$ is not linear over any field of positive characteristic.*

Proof. We can combine Proposition 2 and Theorem 3 to get a contradiction because Dehn twists have infinite order.

□

References

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